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Capacity of the circular plate condenser: analytical solutions for large gaps between the plates

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Abstract
A solution of Love's integral equation (Love E R 1949 *Q. J. Mech. Appl. Math.* **2** 428), which forms the basis for the analysis of the electrostatic field due to two equal circular co-axial parallel conducting plates, is considered for the case when the ratio, τ, of distance of separation to radius of the plates is greater than 2. The kernel of the integral equation is expanded into an infinite series in odd powers of $1/\tau$ and an approximate kernel accurate to $\mathcal{O}(\tau^{-(2N+1)})$ is deduced therefrom by terminating the series after an arbitrary but finite number of terms, N. The approximate kernel is rearranged into a degenerate form and the integral equation with this kernel is reduced to a system of N linear equations. An explicit analytical solution is obtained for $N = 4$ and the resulting analytical expression for the capacity of the circular plate condenser is shown to be accurate to $\mathcal{O}(\tau^{-9})$. Analytical expressions of lower orders of accuracy with respect to $1/\tau$ are deduced from the four-term (i.e., $N = 4$) solution and predictions (of capacity) from the expressions of different orders of accuracy (with respect to $1/\tau$) are compared with very accurate numerical solutions obtained by solving the linear system for large enough N. It is shown that the $\mathcal{O}(\tau^{-9})$ approximation predicts the capacity extremely well for any $\tau \geqslant 2$ and an $\mathcal{O}(\tau^{-3})$ approximation gives, for all practical purposes, results of adequate accuracy for $\tau \geqslant 4$. It is further shown that an approximate solution, applicable for the case of large distances of separation between the plates, due to Sneddon (Sneddon I N 1966 *Mixed Boundary Value Problems in Potential Theory* (Amsterdam: North-Holland) pp 230–46) is accurate to $\mathcal{O}(\tau^{-6})$ for $\tau \geqslant 2$.

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List of symbols

a	radius of the plates
A_n	integrals defined in equation (2.21)

b	distance between the plates
C	capacity of the condenser per unit radius of the plates
$C_p^{(4)}, C_r^{(4)}$	approximations to $C^{(4)}$
$B_n^{(N)}$	unknowns in the system of equations (2.18)
$f(t)$	solution of Love's integral equation
$g_n^{(N)}$	coefficient functions, equation (2.13)
G	function defined in equation (1.3)
I_n	function of τ defined in equation (2.5)
$J_{n,s}^{(N)}$	coefficients in the linear system (2.18), defined in equation (2.19)
$K(t, t')$	modified kernel of Love's integral equation
r, z	radial and axial coordinates
t, t'	variables of Love's integral equation
V	electrostatic potential
V_0	constant potential of the plates

Greek symbols

$\delta^{(1)}, \delta_p^{(4)}, \delta_r^{(4)}$	per cent relative errors in $C^{(1)}, C_p^{(4)}, C_r^{(4)}$
$\Delta^{(N)}$	determinant of the coefficient matrix of the linear system (2.18)
λ	factor that characterizes the charge conditions on the plates
$\mu^{(N)}$	polynomial in $1/\tau$ defined in equation (4.1)
ξ, η	non-dimensional radial and axial coordinates
σ	charge distribution over the plates
τ	(b/a), dimensionless separation between the plates
ϕ	dimensionless electrostatic potential

Superscript

(N)	pertaining to N-term approximation of the kernel

1. Introduction

The analysis of the electrostatic field due to two equal circular parallel co-axial conducting plates is of importance in diverse contexts such as calculation of the capacity of the circular plate condenser [1], measurement of electronic charges in physical experiments involving oil drops [2] and heating of tumours by passing electrical currents between implanted electrodes for the purpose of cancer therapy by hyperthermia [3]. According to Sneddon [1], attempts to solve the circular plate condenser problem go back to Maxwell and Kirchoff who derived an asymptotic approximation for the capacity that is applicable for small separations between the plates. The first systematic and rigorous attack on the problem was probably due to Nicholson [4] who considered two leading cases, namely, to determine the electrostatic field when the plates are charged to (1) *equal* and (2) *equal but opposite* potentials, the potential at infinity being taken to be zero—these two cases, which may be referred to, respectively, as (1) *equally charged plates* and (2) *oppositely charged plates*, are together sufficient to deduce, by superposition, the field due to plates charged to arbitrary (constant) potentials. More specifically, Nicholson expressed the electrostatic potential as a series of spheroidal harmonics in which the coefficients satisfy an infinite system of linear equations and also gave a formula for an explicit solution which was subsequently shown by Love [5] to be erroneous and meaningless. Ignatowsky [6] presented a solution for the potential in the form of a series

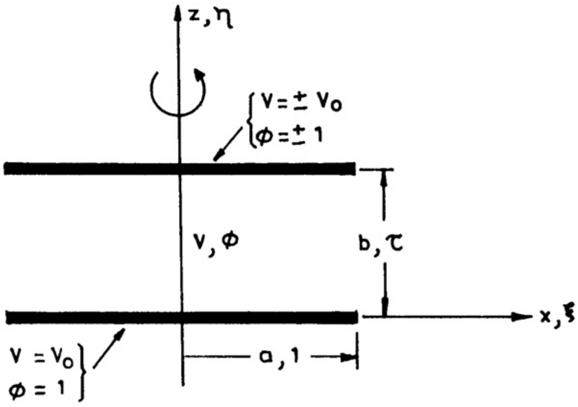

Figure 1. Schematic of the circular plate condenser.

of Legendre functions with the coefficients being determined by an infinite system of linear equations; he also gave an approximate expression for the capacity at small separations that was claimed to be an improvement over that of Kirchoff. Nomura [7] obtained the same series solution as Nicholson but attempted a numerical solution of the associated linear system by iteration without, however, ascertaining the convergence of the iteration process.

Starting tentatively with Nicholson's series solution and the associated linear system, Love [5] worked out an alternate representation of the solution in the form of an integral formula for the electrostatic potential, involving a function that satisfies a Fredholm integral equation of the second kind, and established that it determined uniquely a potential function satisfying all the requirements of the problem. Somewhat simpler derivations of Love's form of the solution were later presented by Cooke [8] and Sneddon [1] by converting the problem to one with mixed boundary conditions and reducing its solution to that of dual integral equations. Referring to figure 1, which depicts the geometric arrangement of the plates, and introducing the non-dimensional quantities

$$\xi = \frac{r}{a}, \qquad \eta = \frac{z}{a}, \qquad \tau = \frac{b}{a} \qquad \text{and} \qquad \phi = \frac{V}{V_0}, \tag{1.1}$$

where a and b are, respectively, the radius and the separation of the plates, V is the electrostatic potential and V_0 is the potential to which the plates are charged, Love's form of the solution, which forms the basis for the sequel, may be expressed as

$$\phi(\xi, \eta) = \frac{1}{\pi} \int_{-1}^{1} \{G(\xi, \eta, t) + \lambda G(\xi, \eta - \tau, t)\} f(t)\, \mathrm{d}t, \tag{1.2}$$

in which

$$G(x, y, t) = \mathrm{Re}\left\{ \frac{1}{x^2 + (y + \mathrm{i}t)^2} \right\}^{\frac{1}{2}} = \left[\frac{(x^2 + y^2 - t^2) + \sqrt{(x^2 + y^2 - t^2)^2 + 4y^2t^2}}{2\{(x^2 + y^2 - t^2)^2 + 4y^2t^2\}} \right]^{\frac{1}{2}} \tag{1.3}$$

and $f(t)$ is an even real function that satisfies the Fredholm integral equation of the second kind

$$f(t) + \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\tau}{\tau^2 + (t - t')^2} f(t')\, \mathrm{d}t' = 1, \qquad |t| \leqslant 1 \tag{1.4}$$

and is related to the charge distribution, $\sigma(\xi)$, over the plates through the simple transformation

$$f(t) = 2\pi \int_{t}^{1} \frac{\xi \sigma(\xi)\, \mathrm{d}\xi}{(\xi^2 - t^2)^{\frac{1}{2}}}, \qquad 0 < t < 1. \tag{1.5}$$

In equations (1.2) and (1.4), λ is a parameter that distinguishes the case of equally charged plates from that of oppositely charged plates and is given by

$$\lambda = \begin{cases} +1, & \text{for equally charged plates,} \\ -1, & \text{for oppositely charged plates.} \end{cases} \tag{1.6}$$

In terms of the function $f(t)$, the capacity, C, of the plates per unit radius is given by

$$C = \frac{1}{\pi} \int_{-1}^{1} f(t)\, \mathrm{d}t. \tag{1.7}$$

Apparently, with this form of the solution, the major task involves the solution of the integral equation (1.4) while the evaluation of the electrostatic potential and the capacity is reduced to simple quadrature. Further, if the interest lies only in evaluating the capacity of the plates then there is no specific need to evaluate the electrostatic potential which, as can be judged from equations (1.2) and (1.3), involves the evaluation of an integral with a complicated integrand.

Unfortunately, the problem has defied a closed-form analytical solution, either via Love's integral equation (1.4) or otherwise, to date. Such a solution was claimed by Atkinson *et al* [9], but their claim was subsequently shown to be false by Hughes [10] and Love [11]. As shown by Love [5] himself, his integral equation can be solved by the method of successive approximations and the resulting Neumann series is convergent for any $\tau > 0$ with a minimum rate of convergence. However, this type of series solution has two serious drawbacks. First, it is not easy, if not altogether impossible, to evaluate the successive terms of the Neumann series in closed form, necessitating some kind of numerical quadrature to obtain the solution which, needless to say, will be purely numerical in nature. Second, the Neumann series is rather slowly convergent for small τ, leading to excessive amount of numerical work. Nevertheless, Bartlett and Corle [12] applied the method of successive approximations to solve Love's equation for the case of oppositely charged plates for three values of τ and confirmed the results by a direct finite-difference relaxation solution of the boundary value problem for the electrostatic potential. Alternatively, starting with Fox and Goodwin [13], a number of studies [14–17] had dealt with the solution of Love's integral equation by direct numerical means—as a matter of fact, over the years, Love's equation had served as one of the standard problems for testing the efficacy of various numerical approaches for solving Fredholm integral equations of the second kind. In general, most of these direct numerical methods were found to face problems of ill-conditioning, excessive computational effort and low accuracy for $\tau \ll 1$. Hutson [3], Bartlett and Corle [12] and Sloggett *et al* [18] considered the numerical solution of the boundary value problem (for the electrostatic potential) *per se* by finite-difference relaxation methods, which were also found to be not very efficient for $\tau \ll 1$, requiring excessively large computational effort coupled with poor accuracy for the fringing field around the edges of the plates.

In view of the non-availability of exact closed-form analytical solutions and the inherent difficulties and limitations associated with series type or numerical type of solutions, over the years there have been many attempts to obtain useful analytical approximations for the capacity of the parallel plate condenser. A majority of these studies were concerned with finding analytical approximations for the capacity for small separations between the plates (i.e., $\tau \ll 1$), with the approximate solutions of Kirchoff and Ignatowsky serving as benchmark results. Thus, for oppositely charged plates, Pólya and Szegö [19] used a variational technique to get an asymptotic lower bound for the capacity, accurate to $\mathcal{O}(\tau)$, which coincided with the approximate formula of Ignatowsky. Following a similar variational approach, Noble [20] had reduced the evaluation of the capacity to the minimization of a functional with the solution, f, of Love's integral equation (1.4) as the candidate function; for oppositely charged plates, a trial value of f based on a constant charge density (equation (1.5)) over the plates led to a capacity estimate equivalent to that of Pólya and Szegö. Following an idea by Maxwell whereby the problem could be reduced to a two-dimensional one and solved by a conformal mapping technique, Cooke [21] had derived an asymptotic expression for the capacity that predicted slightly lower values than the expressions of Kirchoff and Ignatowsky. By repeated application of an integral formula due to Kac and Pollard [22] for an approximate solution of any inhomogeneous Fredholm integral equation of the second kind with Love's kernel, Hutson [23] derived a uniformly valid approximate solution for Love's integral equation (1.4) and rigorously established that Kirchoff's solution was the correct asymptotic representation for the capacity of the oppositely charged parallel plate condenser for $\tau \to 0$. Leppington and Levine [24] derived an integral equation of the first kind for the distribution of the electrostatic potential off the plates and utilized it to obtain an approximation for the capacity when τ is small, reproducing, in the process, the result of Kirchoff in addition to providing an explicit estimate of the order of the error with respect to τ in Kirchoff's expression. Sloggett *et al* [18] had analytically derived a first-order estimate for the capacity, which was further improved upon by including an empirically adjusted second-order term, and evaluated the accuracies of the various approximate expressions in the light of the numerical results of Cooke [21] and Bartlett and Corle [12]. For the case of equally charged plates, Leppington and Levine [25] had derived an asymptotic estimate for the capacity valid for $\tau \to 0$ by constructing a uniformly accurate distribution of the electrostatic potential over the whole plane containing one of the plates. Atkinson and Leppington [26] had reproduced the result of [25] by solving the appropriate Love's equation (1.4) with the help of matched asymptotic expansions.

Surprisingly, attempts to obtain useful analytical approximations for the case of large separations between the plates have been rather scanty. In fact, the only such attempt this author is aware of appears to be the one presented in [1], where, based on the variational approach of Noble [20], an analytical solution is obtained for the capacity (equation (8.1.42) of [1]) under the physically plausible assumption that, for large separations, the charge distribution on either of the plates is same as that existing on an isolated plate raised to a uniform potential. Though the ensuing solution seems to give acceptable answers for large distances between the plates, it must be realized that, in mathematical terms, a different problem, namely one with specified charge distribution on the plates is solved rather than the original problem of plates raised to constant potential. As such, there seem to be no approximate analytical solutions, applicable for large distances of separation between the plates, based on Love's equation and this paper is devoted to this aspect.

In the present work, by exploiting the even function property of $f(t)$, the kernel of Love's equation is rearranged to limit the range of integration to 0–1. The modified kernel is then expanded into an infinite series in odd powers of $1/\tau$ for the specific case of $\tau > 2$. Approximate kernels, accurate to $\mathcal{O}(\tau^{-(2N+1)})$, are deduced therefrom by terminating the series after a finite number, say N, of terms. The approximate kernel is then rearranged into a degenerate form and the solution of the integral equation with this approximate kernel is reduced to a well-conditioned diagonally dominant system of N linear algebraic equations. (The exact solution, for this case of $\tau > 2$, is recovered in the limit of $N \to \infty$.) An explicit analytical solution is obtained for $N = 4$ and the resulting analytical expression for the capacity of the circular plate condenser is shown to be accurate to $\mathcal{O}(\tau^{-9})$. Analytical expressions of lower orders of accuracy (with respect to $1/\tau$) are deduced from the four-term (i.e., $N = 4$) solution and predictions (of capacity) from the approximate solutions of different orders of accuracy (with respect to $1/\tau$) are compared with very accurate numerical solutions obtained by solving the linear system for large enough N to ensure convergence to within a prescribed error tolerance. It is shown that the $\mathcal{O}(\tau^{-9})$ approximation provides an exceedingly good approximation to the capacity for any $\tau \geqslant 2$ and an $\mathcal{O}(\tau^{-3})$ approximation gives, for all practical purposes, results of adequate accuracy for any $\tau \geqslant 4$. It is further shown that the approximate solution of Sneddon, mentioned in the previous paragraph, is accurate to $\mathcal{O}(\tau^{-6})$ for $\tau \geqslant 2$.

The reduction of the problem to a linear system of algebraic equations for the case of $\tau > 2$ and the accurate analytical approximations for the capacity derived therefrom are the major contributions of this study. The kind of solution procedure developed here does not seem to have been reported earlier and it is hoped that the method will prove useful and find application in solving integral equations with kernels similar in structure to that of Love, as appears to be the case in some contact problems in heat transfer [27]. We present below in section 2 the analytical developments leading to the linear system of algebraic equations followed by the explicit analytical solution for a four-term approximation of the kernel in section 3, simplification and assessment of the analytical approximations in sections 4 and 5, and, finally, the main conclusions of this study in section 6.

2. Reduction to system of linear equations for $\tau > 2$

Since $f(t)$ is an even function of t, Love's integral equation may be recast as

$$f(t) + \int_0^1 K(t, t') f(t')\, \mathrm{d}t' = 1, \qquad |t| \leqslant 1, \tag{2.1}$$

where, now, the kernel is given by

$$K(t, t') = \left(\frac{\lambda}{\pi}\right)\left[\frac{\tau}{\tau^2 + (t + t')^2} + \frac{\tau}{\tau^2 + (t - t')^2}\right]. \tag{2.2}$$

Rewriting the above expression for the kernel in the form

$$K(t, t') = \left(\frac{\lambda}{\pi\tau}\right)\left[\left\{1 + \left(\frac{t + t'}{\tau}\right)^2\right\}^{-1} + \left\{1 + \left(\frac{t - t'}{\tau}\right)^2\right\}^{-1}\right], \tag{2.3}$$

formally expanding the expressions in the curly brackets as binomial series in powers of $\{(t \pm t')/\tau\}^2$ and rearranging, we have

$$K(t, t') = -\sum_{n=1}^{\infty} (-1)^n \frac{I_n}{2}\{(t + t')^{2(n-1)} + (t - t')^{2(n-1)}\}, \tag{2.4}$$

where

$$I_n = \left(\frac{2\lambda}{\pi}\right)\tau^{-(2n-1)}, \qquad n = 1, 2, \ldots. \tag{2.5}$$

Noting that

$$\left.\begin{aligned} \max|t + t'| &= 2 \\ \max|t - t'| &= 1 \end{aligned}\right\} \qquad \text{for} \quad 0 < t \quad \text{and} \quad t' < 1, \tag{2.6}$$

it may be seen that equation (2.4) is a valid representation of the kernel for (and only for) any value of $\tau > 2$, for, then

$$\max\left|\frac{t \pm t'}{\tau}\right| < 1, \tag{2.7}$$

thereby ensuring the absolute convergence of the binomial series on (0, 1). Further, in view of the fact that $I_n \sim \tau^{-(2n-1)}$, it may be easily recognized that equation (2.4) is the expansion of the kernel as a uniformly convergent series in odd powers of $1/\tau$ for $\tau > 2$. Consequently, the kernel may be approximated to any desired (odd) degree of accuracy with respect to $1/\tau$ by terminating the series after an appropriate number of terms. Thus, by retaining only the first N terms of the series and distinguishing the resulting approximate kernel by the superscript (N), we have

$$K(t,t') = K^{(N)}(t,t') + \mathcal{O}(\tau^{-(2N+1)}). \tag{2.8}$$

Expanding the terms $(t \pm t')^{2(n-1)}$ in equation (2.4) with the help of the binomial theorem and rearranging, the approximate kernel, $K^{(N)}(t,t')$, may be expressed as

$$K^{(N)}(t,t') = -\sum_{n=1}^{N}(-1)^n \frac{I_n}{2}\sum_{k=0}^{2(n-1)}\binom{2(n-1)}{k}\{t'^{2(n-1)-k} + (-t')^{2(n-1)-k}\}t^k, \tag{2.9}$$

in which

$$\binom{m}{k} = \frac{m!}{(m-k)!k!} \tag{2.10}$$

stands for the usual binomial coefficient. Noting that the term in the curly brackets in equation (2.9) is non-zero only if $\{2(n-1)-k\}$ is an even number, it follows that, since $2(n-1)$ is even, k must be even. As a consequence, equation (2.9) may be expressed as

$$K^{(N)}(t,t') = -\sum_{n=1}^{N}(-1)^n I_n \sum_{l=0}^{(n-1)}\binom{2(n-1)}{2l} t'^{2(n-1-l)}t^{2l}. \tag{2.11}$$

Unfolding the above series, collecting together terms containing like powers of t and rearranging, $K^{(N)}(t,t')$ may finally be reduced to the degenerate form

$$K^{(N)}(t,t') = -\sum_{n=1}^{N} g_n^{(N)}(t')t^{2(n-1)}, \tag{2.12}$$

in which the coefficient functions, $g_n^{(N)}(t')$, are even-degree polynomials in t' given by

$$g_n^{(N)}(t') = \sum_{l=n}^{N}(-1)^l I_l \binom{2(l-1)}{2(n-1)} t'^{2(l-n)}, \qquad n = 1, 2, \ldots, N. \tag{2.13}$$

Here the superscript (N) and the subscript n are used to signify the fact that $g_n^{(N)}(t')$ is the coefficient of the nth term in the expansion of the (approximate) kernel, $K^{(N)}(t,t')$, as a (finite) series in ascending powers of t^2. Apparently, $g_n^{(N)}(t') \neq g_n^{(N-1)}(t')$. However, the two may be related to each other by the relation

$$g_n^{(N)}(t') = g_n^{(N-1)}(t') + (-1)^N I_N \binom{2(N-1)}{2(n-1)} t'^{2(N-n)}, \qquad n = 1, 2, \ldots, N-1. \tag{2.14}$$

Augmenting this equation with the relation

$$g_N^{(N)}(t') = (-1)^N I_N, \tag{2.15}$$

it may be seen that, starting with $g_1^{(1)}(t') = -I_1$, all $g_n^{(N)}(t')$, $n = 1, 2, \ldots, N$; $N = 2, 3, \ldots,$ may be obtained recursively for any arbitrary N.

The fact that $K^{(N)}(t,t')$ (equation (2.12)) is degenerate may be conveniently exploited to obtain approximate solutions to the problem. Towards this end, let $f^{(N)}(t)$ be the solution of the integral equation (2.1) when the original kernel, $K(t,t')$, is replaced by the approximate kernel, $K^{(N)}(t,t')$. Since the latter is accurate to $\mathcal{O}(\tau^{-(2N+1)})$, the solution thereof may also be expected to have the same order of accuracy with respect to τ.[1] Replacing, therefore, $K(t, t')$ in equation (2.1) by $K^{(N)}(t, t')$ from equation (2.12), interchanging the order of integration and summation in the resulting expression and rearranging, we have

$$f^{(N)}(t) = 1 + \sum_{n=1}^{N} B_n^{(N)} t^{2(n-1)}, \tag{2.16}$$

where

$$B_n^{(N)} = \int_0^1 g_n^{(N)}(t') f^{(N)}(t')\, \mathrm{d}t', \qquad n = 1, 2, \ldots, N, \tag{2.17}$$

are, as yet unknown, constants. Substituting for $f^{(N)}(t')$ from equation (2.16) and rearranging, the above equation may be easily reduced to the linear system

$$B_n^{(N)} - \sum_{s=1}^{N} J_{n,s}^{(N)} B_s^{(N)} = J_{n,1}^{(N)}, \qquad n = 1, 2, \ldots, N, \tag{2.18}$$

in which

$$\begin{aligned} J_{n,s}^{(N)} &= \int_0^1 g_n^{(N)}(t') t'^{2(s-1)}\, \mathrm{d}t' \\ &= \sum_{l=n}^{N} (-1)^l I_l \binom{2(l-1)}{2(n-1)} \frac{1}{2(s+l-n)-1}, \qquad s = 1, 2, \ldots, N; \quad n = 1, 2, \ldots, N, \end{aligned} \tag{2.19}$$

are known quantities dependent only on I_l, $l = n, \ldots, N$, and, hence, on τ (>2) (equation (2.5)).

Substituting $f^{(N)}(t)$ (equation (2.16)) in place of $f(t)$ in equation (1.2) and noting that both $G(x, y, t)$ (equation (1.3)) and $f^{(N)}(t)$ are even functions of t, the dimensionless electrostatic potential, $\phi^{(N)}(\xi, \eta)$, corresponding to $K^{(N)}(t, t')$ may be expressed as

$$\phi^{(N)}(\xi, \eta) = \frac{2}{\pi}\left[A_1 + \sum_{n=1}^{N} B_n^{(N)} A_n\right], \tag{2.20}$$

where

$$A_n = \int_0^1 \{G(\xi, \eta, t) + \lambda G(\xi, \eta - \tau, t)\} t^{2(n-1)}\, \mathrm{d}t, \qquad n = 1, 2, \ldots, N. \tag{2.21}$$

Finally, replacing $f(t)$ in equation (1.7) by $f^{(N)}(t)$ from equation (2.16) and denoting by $C^{(N)}$ the resulting approximation to the capacity, we have

$$C^{(N)} = \left(\frac{2}{\pi}\right)\left[1 + \sum_{n=1}^{N} \frac{B_n^{(N)}}{(2n-1)}\right]. \tag{2.22}$$

It may be expected that $\phi^{(N)}(\xi, \eta)$ as well as $C^{(N)}$ will have the same order of accuracy with respect to $1/\tau$ as $f^{(N)}(t)$, that is, $|C - C^{(N)}| \sim \mathcal{O}(\tau^{-(2N+1)})$ and $|\phi - \phi^{(N)}| \sim \mathcal{O}(\tau^{-(2N+1)})$.

[1] In fact, this point may be established rather rigorously. Since, from equation (2.8), $K(t, t') - K^{(N)}(t, t') = F(t, t')/\tau^{2N+1} + \mathcal{O}(\tau^{-(2N+3)})$, it follows that $\int_0^1 |K(t, t') - K^{(N)}(t, t')|\, \mathrm{d}t' \sim \mathcal{O}(\tau^{-(2N+1)})$ and, by virtue of a theorem by Kantorovich [28], $|f(t) - f^{(N)}(t)| \sim \mathcal{O}(\tau^{-(2N+1)})$.

3. Solution for $N = 4$

From the foregoing it is evident that, for any arbitrary order of approximation (with respect to $1/\tau$) of the kernel as characterized by the number N, the problem is reduced to solving the system of linear equations (2.18) for $B_n^{(N)}, n = 1, 2, \ldots, N$, and determining the corresponding potential field and associated capacity from equations (2.20) and (2.22), respectively. As can be easily inferred from the definition of $K^{(N)}(t, t')$ (equation (2.9)), the solution (that is, $f^{(N)}(t)$) so obtained contains all the lower order solutions (that is, $f^{(N-M)}(t), M = 1, 2, \ldots, N - 1$) and, provided $B_n^{(N)}, n = 1, 2, \ldots, N$, can be evaluated analytically, the latter may be recovered, for any $M < N$, by setting $I_{N-L+1} = 0, L = 1, 2, \ldots, M$, in the former. Solutions of any desired accuracy may be obtained by a suitable choice of N—in fact, a sequence of solutions may be generated for a sequence of increasing values of N until a desired level of accuracy is attained. It is not difficult to recognize that this process is akin to solving a sequence of approximate problems, each of which is a better approximation to the original problem than its predecessors. The major difficulty in carrying out this process lies in the fact that, as N is increased, all $J_{n,s}^{(N)}, s = 1, 2, \ldots, N; n = 1, 2, \ldots, N$, have to be re-evaluated since $J_{n,s}^{(N)} \neq J_{n,s}^{(M)}, s = 1, 2, \ldots, M; n = 1, 2, \ldots, M$, for $N > M$. However, starting with any $J_{n,s}^{(M)}, s = 1, \ldots, M; n = 1, \ldots, M$, the requisite $J_{n,s}^{(N)}, s = 1, \ldots, N; n = 1, \ldots, N$, may be easily calculated recursively for any $N > M$ with the help of the relations

$$J_{n,s}^{(N)} = J_{n,s}^{(N-1)} + (-1)^N I_N \binom{2(N-1)}{2(n-1)} \frac{1}{\{2(s+N-n)-1\}},$$
$$s = 1, 2, \ldots, N-1; \quad n = 1, 2, \ldots, N-1, \tag{3.1}$$

$$J_{n,N}^{(N)} = \sum_{l=n}^{N} (-1)^l I_l \binom{2(l-1)}{2(n-1)} \frac{1}{\{2(l+N-n)-1\}}, \qquad n = 1, 2, \ldots, N-1, \tag{3.2}$$

$$J_{N,s}^{(N)} = (-1)^N \frac{I_N}{(2s-1)}, \qquad s = 1, 2, \ldots, N, \tag{3.3}$$

which are a consequence of equations (2.14) and (2.19).

An alternative to the foregoing strategy of solving the problem may be imagined based on the observation that the exact solution is obtained in the limit as $N \to \infty$. In this case, the linear system (2.18) is transformed to an infinite system with $J_{n,s}^{(N)}$ replaced by $J_{n,s}^{(\infty)}, s = 1, 2, \ldots;$ $n = 1, 2, \ldots,$ and $B_n^{(N)}$ replaced by $B_n^{(\infty)}, n = 1, 2, \ldots.$ Since the only practical way of solving this infinite system is by the method of reduction, that is, by means of a passage to the limit in the solution of the finite system obtained from the infinite system by discarding all equations and unknowns beyond a certain number, the task is reduced to one of obtaining a sequence of solutions to a sequence of reduced systems of increasing size until a desired level of accuracy is attained. Though this strategy is not very much different from the previous one, it has the advantage that, as the size of the reduced system is increased, the only coefficients that need to be evaluated afresh are those corresponding to the added unknowns and those contained in the added equations. In the final analysis, both strategies should lead to the same answers and there is hardly anything to prefer one over the other, especially if solutions are sought by numerical means. However, when analytic solutions are sought, the previous strategy is to be preferred since it involves fewer manipulations due to the fact that $J_{n,s}^{(N)}$ is made up of finite number of terms unlike $J_{n,s}^{(\infty)}$ which contains an infinite number of terms. Moreover, it possesses the advantage of giving a precise idea of the order of the error involved in the solution.

Whatever strategy is finally adopted, it is important to realize that it is not easy to solve the linear system (2.18) analytically unless the size is rather small, say of the order of 4 or less, and recourse has often to be made to numerical means. Consequently, we confine ourselves to obtaining an analytical solution for a four-term approximation of the kernel, i.e., $N = 4$, this choice being dictated by a compromise between the labour involved and the accuracy attained. It may be noted that, for any $\tau > 2$, I_n decreases rather rapidly with increasing n and, as is indeed borne out by comparison with exact numerical solutions to be presented later on, it is not unreasonable to expect fairly accurate solutions with only a few terms. For this case of $N = 4$, we have

$$\begin{Bmatrix} g_1^{(4)}(t') \\ g_2^{(4)}(t') \\ g_3^{(4)}(t') \\ g_4^{(4)}(t') \end{Bmatrix} = \begin{bmatrix} -1 & t'^2 & -t'^4 & t'^6 \\ 0 & 1 & -6t'^2 & 15t'^4 \\ 0 & 0 & -1 & 15t'^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{Bmatrix}. \tag{3.4}$$

The corresponding $J_{n,s}^{(N)}$ are given by

$$\begin{Bmatrix} J_{1,s}^{(4)} \\ J_{2,s}^{(4)} \\ J_{3,s}^{(4)} \\ J_{4,s}^{(4)} \end{Bmatrix} = \begin{bmatrix} \frac{-1}{(2s-1)} & \frac{1}{(2s+1)} & \frac{-1}{(2s+3)} & \frac{1}{(2s+5)} \\ 0 & \frac{1}{(2s-1)} & \frac{-6}{(2s+1)} & \frac{15}{(2s+3)} \\ 0 & 0 & \frac{-1}{(2s-1)} & \frac{15}{(2s+1)} \\ 0 & 0 & 0 & \frac{1}{(2s-1)} \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{Bmatrix}, \qquad s = 1, \ldots, 4. \tag{3.5}$$

The linear system of equations (2.18) is solved by Kramer's rule for $N = 4$ leading, after very tedious but straightforward algebra, to the solution

$$B_4^{(4)} = \frac{1}{\Delta^{(4)}} \left(I_4 + \frac{8}{15} I_3 I_4 - \frac{16}{7} I_4^2 - \frac{256}{2205} I_4^3 \right), \tag{3.6}$$

$$B_3^{(4)} = \frac{-1}{\Delta^{(4)}} \left(I_3 - 5I_4 - \frac{4}{3} I_2 I_4 + \frac{8}{15} I_3^2 - \frac{8}{7} I_3 I_4 + \frac{16}{21} I_4^2 - \frac{256}{1617} I_4^3 \right), \tag{3.7}$$

$$B_2^{(4)} = \frac{1}{\Delta^{(4)}} \left(I_2 - 2I_3 + 3I_4 - \frac{8}{7} I_2 I_4 + \frac{16}{35} I_3^2 - \frac{152}{105} I_3 I_4 + \frac{656}{231} I_4^2 - \frac{256}{4851} I_4^3 \right), \tag{3.8}$$

$$\begin{aligned} B_1^{(4)} = \frac{-1}{\Delta^{(4)}} \bigg(& I_1 - \frac{I_2}{3} + \frac{I_3}{5} - \frac{I_4}{7} + \frac{8}{15} I_1 I_3 - \frac{16}{7} I_1 I_4 - \frac{4}{45} I_2^2 + \frac{16}{105} I_2 I_3 \\ & - \frac{4}{315} I_2 I_4 - \frac{184}{1575} I_3^2 + \frac{1048}{3465} I_3 I_4 - \frac{4880}{21\,021} I_4^2 - \frac{256}{2205} I_1 I_4^2 \\ & + \frac{512}{33\,075} I_2 I_3 I_4 - \frac{512}{165\,375} I_3^3 - \frac{512}{24\,255} I_2 I_4^2 + \frac{1024}{121\,275} I_3^2 I_4 \\ & - \frac{4096}{315\,315} I_3 I_4^2 + \frac{17\,152}{1324\,323} I_4^3 + \frac{65\,536}{1529\,593\,065} I_4^4 \bigg), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \Delta^{(4)} = 1 + I_1 - \frac{2}{3} I_2 + \frac{8}{5} I_3 - \frac{32}{7} I_4 + \frac{8}{15} I_1 I_3 - \frac{16}{7} I_1 I_4 - \frac{4}{45} I_2^2 + \frac{16}{105} I_2 I_3 \\ + \frac{32}{315} I_2 I_4 - \frac{256}{1575} I_3^2 + \frac{1664}{3465} I_3 I_4 - \frac{17\,152}{21\,021} I_4^2 - \frac{256}{2205} I_1 I_4^2 \end{aligned}$$

$$+\frac{512}{33\,075}I_2I_3I_4-\frac{512}{165\,375}I_3^3-\frac{512}{24\,255}I_2I_4^2+\frac{1024}{121\,275}I_3^2I_4$$
$$-\frac{4096}{315\,315}I_3I_4^2+\frac{20\,480}{1324\,323}I_4^3+\frac{65\,536}{1529\,593\,065}I_4^4. \tag{3.10}$$

Substituting the above values of $B_n^{(4)}, n = 1, \ldots, 4$, into equation (2.22), we have

$$C^{(4)}=\frac{2}{\pi\,\Delta^{(4)}}\left(1+\frac{8}{15}I_3-\frac{16}{7}I_4-\frac{256}{2205}I_4^2\right). \tag{3.11}$$

Substituting for $I_n, n = 1, \ldots, 4$, from equation (2.5), we finally have

$$C^{(4)}=\frac{2}{\pi\,\Delta^{(4)}}\left\{1+\frac{\lambda}{\pi}\left(\frac{16}{15\tau^5}-\frac{32}{7\tau^7}\right)-\frac{\lambda^2}{\pi^2}\frac{1024}{2205\tau^{14}}\right\}, \tag{3.12}$$

where, now,

$$\Delta^{(4)}=1+\frac{\lambda}{\pi}\left(\frac{2}{\tau}-\frac{4}{3\tau^3}+\frac{16}{5\tau^5}-\frac{64}{7\tau^7}\right)+\frac{\lambda^2}{\pi^2}\left(\frac{16}{9\tau^6}-\frac{128}{15\tau^8}-\frac{128}{525\tau^{10}}\right.$$
$$\left.+\frac{6656}{3465\tau^{12}}-\frac{68\,608}{21\,021\tau^{14}}\right)-\frac{\lambda^3}{\pi^3}\left(\frac{137\,216}{165\,375\tau^{15}}+\frac{4096}{40\,425\tau^{17}}+\frac{32\,768}{315\,315\tau^{19}}\right.$$
$$\left.-\frac{163\,840}{1\,324\,323\tau^{21}}\right)+\frac{\lambda^4}{\pi^4}\left(\frac{1048\,576}{1529\,593\,065\tau^{28}}\right). \tag{3.13}$$

As pointed out earlier, the lower order capacities, $C^{(4-M)}$, $M = 1, 2, 3$, are obtained by setting $I_{4-L+1} = 0, L = 1, \ldots, M$, in the above equation. We thus have, after simplification, the relations

$$C^{(3)}=\frac{2}{\pi}\left(1+\frac{\lambda}{\pi}\frac{16}{15\tau^5}\right)\bigg/\left\{1+\frac{\lambda}{\pi}\left(\frac{2}{\tau}-\frac{4}{3\tau^3}+\frac{16}{5\tau^5}\right)\right.$$
$$\left.+\frac{\lambda^2}{\pi^2}\left(\frac{16}{9\tau^6}+\frac{64}{105\tau^8}-\frac{1024}{1575\tau^{10}}\right)-\frac{\lambda^3}{\pi^3}\left(\frac{2048}{165\,375\tau^{15}}\right)\right\}, \tag{3.14}$$

$$C^{(2)}=\left(\frac{2}{\pi}\right)\bigg/\left\{1+\frac{\lambda}{\pi}\left(\frac{2}{\tau}-\frac{4}{3\tau^3}\right)-\frac{\lambda^2}{\pi^2}\frac{16}{45\tau^6}\right\}, \tag{3.15}$$

$$C^{(1)}=\left(\frac{2}{\pi}\right)\bigg/\left(1+\frac{\lambda}{\pi}\frac{2}{\tau}\right). \tag{3.16}$$

It is not easy to obtain similar closed-form expressions for $\phi^{(N)}$, $N = 1, \ldots, 4$ (equation (2.20)), since it appears to be rather very difficult, if not altogether impossible, to evaluate the integrals defining the various A_n, $n = 1, \ldots, 4$ (equation (2.21)), analytically.

4. Approximations to $C^{(4)}$, etc

It may be seen from equations (3.12)–(3.16) that the expressions for the capacity of the condenser become increasingly complex and unwieldy with increasing order of approximation of the kernel with respect to $1/\tau$. More importantly, except for $C^{(1)}$, the expression for the capacity at a given order of approximation contains, in both the numerator and the denominator, terms involving powers of $1/\tau$ larger than the likely order of accuracy with respect to $1/\tau$ of the approximation itself in relation to the exact value, C. However, notwithstanding the presence of such higher order terms, the approximation $C^{(N)}$ is accurate to only $\mathcal{O}(\tau^{-(2N+1)})$ as will be shown below. Further, the coefficients of these higher order terms are orders of magnitude smaller than those of the lower order terms. This observation coupled with the fact that $C^{(N)}$ is accurate to $\mathcal{O}(\tau^{-(2N+1)})$ may be exploited to effect substantial simplification in the above expressions for $C^{(4)}$, etc. To see clearly what is involved, we shall first recast $C^{(N)}$ in the form

$$C^{(N)} = \frac{2}{\pi}\frac{1}{\mu^{(N)}}, \qquad N = 1, \ldots, 4, \tag{4.1}$$

where $\mu^{(N)}$ is a polynomial in $1/\tau$ obtained by dividing the denominator in the expression for $C^{(N)}$ by the corresponding numerator. Thus, for the expressions given by equations (3.12)–(3.16), we have

$$\mu^{(1)} = 1 + \frac{2\lambda}{\pi\tau}, \tag{4.2}$$

$$\mu^{(2)} = 1 + \frac{\lambda}{\pi}\left(\frac{2}{\tau} - \frac{4}{3\tau^3}\right) - \frac{\lambda^2}{\pi^2}\frac{16}{45\tau^6}, \tag{4.3}$$

$$\mu(3) = 1 + \frac{\lambda}{\pi}\left(\frac{2}{\tau} - \frac{4}{3\tau^3} + \frac{32}{15\tau^5}\right) - \frac{\lambda^2}{\pi^2}\left(\frac{16}{45\tau^6} - \frac{128}{63\tau^8} + \frac{512}{175\tau^{10}}\right) + \mathcal{O}(\tau^{-11}), \tag{4.4}$$

$$\mu^{(4)} = 1 + \frac{\lambda}{\pi}\left(\frac{2}{\pi} - \frac{4}{3\tau^3} + \frac{32}{15\tau^5} - \frac{32}{7\tau^7}\right) - \frac{\lambda^2}{\pi^2}\left(\frac{16}{45\tau^6} - \frac{128}{63\tau^8} + \frac{13\,658}{1575\tau^{10}}\right) + \mathcal{O}(\tau^{-11}). \tag{4.5}$$

It may be noted that it is possible, in principle at least, to express the capacity corresponding to any arbitrary order of approximation of Love's kernel as the reciprocal of a polynomial in $1/\tau$. By the same token, increasing the order of approximation indefinitely the exact solution for the capacity, C, may be expressed as the reciprocal of an infinite series in ascending powers of $1/\tau$. What is more important, it may be seen that equations (4.2)–(4.5) provide a rational basis for ascertaining the order of accuracy with respect to $1/\tau$ of $\mu^{(N)}$, $N = 1, 2, \ldots$, and, hence, the capacity $C^{(N)}$, $N = 1, 2, \ldots$.

From a close inspection of equations (4.2)–(4.5), it may be seen that two successive approximations, say $\mu^{(N)}$ and $\mu^{(N+1)}$, $N = 1, 2$ or 3, will have the same common terms up to $\mathcal{O}(\tau^{-2N})$ and differ from one another only in terms of $\mathcal{O}(\tau^{-(2N+1)})$ and higher; that is, $|\mu^{(N+1)} - \mu^{(N)}| \sim \mathcal{O}(\tau^{-(2N+1)})$, $N = 1, 2, \ldots$. Generalizing this observation somewhat, it may be argued that as the order of approximation is increased arbitrarily, say from N to $(N + M)$, the resulting $\mu^{(N+M)}$ will differ from $\mu^{(N)}$ only through the addition of new terms of $\mathcal{O}(\tau^{-(2N+1)})$ and higher to the already existing terms in the latter and the lower order terms up to $\mathcal{O}(\tau^{-2N})$ will be common for both—compare, for instance, equations (4.2)–(4.4) with equation (4.5). By extension, it follows that $\mu^{(N)}$ will share the same common terms with the exact solution, $\mu^{(\infty)}$, up to $\mathcal{O}(\tau^{-2N})$ and deviate from the latter only in higher order terms of $\mathcal{O}(\tau^{-(2N+1)})$ and above, thereby indicating that it (i.e., $\mu^{(N)}$) is indeed accurate only to $\mathcal{O}(\tau^{-(2N+1)})$. Writing $\mu^{(N)} = \mu^{(\infty)} - (\mu^{(\infty)} - \mu^{(N)})$, substituting into equation (4.1) and making use of the binomial theorem, it may be seen that

$$C^{(N)} = \frac{2}{\pi}\frac{1}{\mu^{(\infty)}} + \mathcal{O}\left(\frac{\mu^{(\infty)} - \mu^{(N)}}{\mu^{(\infty)}}\right), \tag{4.6}$$

where use has been made of the observation that, for $\tau > 2$, $\mu^{(N)}$, $N = 1, 2, \ldots$, is of $\mathcal{O}(1)$. Since $(\mu^{(\infty)} - \mu^{(N)})$ is of $\mathcal{O}(\tau^{-(2N+1)})$, it follows that

$$|C - C^{(N)}| \sim \mathcal{O}(\tau^{-(2N+1)}), \tag{4.7}$$

thereby establishing that $C^{(N)}$, $N = 1, \ldots, 4$, is accurate to $\mathcal{O}(\tau^{-(2N+1)})$.

However, as can be judged from equations (4.3)–(4.5), though accurate only to $\mathcal{O}(\tau^{-(2N+1)})$, $\mu^{(N)}$ still contains many higher order terms in $1/\tau$ whose coefficients are orders of magnitude smaller than those of the lower order terms up to $\mathcal{O}(\tau^{-2N})$. Consequently, it seems reasonable to approximate $C^{(N)}$, $N = 2, 3, \ldots$, by ignoring in $\mu^{(N)}$ terms involving powers of $1/\tau$ larger than $2N$ and, since the errors so incurred are of a higher order of smallness than the leading error term between $\mu^{(\infty)}$ and $\mu^{(N)}$, the so approximated expression of $C^{(N)}$ may be expected to be as good an approximation to the exact solution, C, as the original expression itself. Applying this philosophy, we may therefore approximate $C^{(4)}$ as

$$C^{(4)} \approx C_p^{(4)} = \frac{2}{\pi}\left[1 + \frac{\lambda}{\pi}\left(\frac{2}{\tau} - \frac{4}{3\tau^3} + \frac{32}{15\tau^5} - \frac{32}{7\tau^7}\right) - \frac{\lambda^2}{\pi^2}\left(\frac{16}{45\tau^6} - \frac{128}{63\tau^8}\right)\right]^{-1}. \tag{4.8}$$

Similar approximate expressions may be written for $C^{(2)}$ and $C^{(3)}$ as well without much difficulty.

In the light of the arguments leading to it, it is apparent that the expression inside the square brackets on the right-hand side of equation (4.8) constitutes the first seven terms in the probable representation of the reciprocal of the exact value of the capacity—more precisely, the quantity $(2/(\pi C))$—as an infinite series in ascending powers of $1/\tau$ and is accurate to $\mathcal{O}(\tau^{-9})$. The first term corresponds to the capacity of a single plate raised to a uniform potential (i.e., $C = (2/\pi)$) and the subsequent terms account for the presence of the second plate at a distance of τ. As is to be excepted, the effect of the plate separation becomes stronger as τ decreases towards 2 and is reflected by the necessity of taking into account an increasing number of terms in the series.

Surprisingly, not only $C^{(N)}$ as such but both its numerator (say, $P^{(N)}$) and its denominator (say, $Q^{(N)}$) also seem to be individually accurate to $\mathcal{O}(\tau^{-(2N+1)})$, that is, if $C^{(N)}$ tends, in the limit as $N \to \infty$, to $C = P/Q$, where both P and Q are likely to be infinite series in powers of $1/\tau$, then both $|P - P^{(N)}|$ and $|Q - Q^{(N)}|$ are $\mathcal{O}(\tau^{-(2N+1)})$. That it is so may be easily ascertained by comparing the numerators as well as the denominators in equations (3.12)–(3.16) separately and invoking arguments similar to those employed above in establishing the order of accuracy of $C^{(N)}$. An important consequence of this observation is that an alternate kind of approximation may be effected in the expression for $C^{(N)}$. Since $P^{(N)}$ and $Q^{(N)}$ are individually accurate to $\mathcal{O}(\tau^{-(2N+1)})$ despite the presence of higher order terms and since the coefficients of these higher order terms are orders of magnitude smaller than those of the lower order terms up to $\mathcal{O}(\tau^{-2N})$, it seems possible to ignore the higher order terms in both $P^{(N)}$ and $Q^{(N)}$ without entailing any significant loss in accuracy. Applying this line of reasoning, we may approximate $C^{(4)}$ by the expression

$$C^{(4)} \approx C_r^{(4)} = \frac{2}{\pi}\left[\frac{1 + \frac{\lambda}{\pi}\left(\frac{16}{15\tau^5} - \frac{32}{7\tau^7}\right)}{1 + \frac{\lambda}{\pi}\left(\frac{2}{\tau} - \frac{4}{3\tau^3} + \frac{16}{5\tau^5} - \frac{64}{7\tau^7}\right) + \frac{\lambda^2}{\pi^2}\left(\frac{16}{9\tau^6} - \frac{128}{15\tau^8}\right)}\right]. \tag{4.9}$$

Similar expressions may be derived for $C^{(2)}$ and $C^{(3)}$ also.

It should be noted that, though all of them have equal status insofar as the order of accuracy with respect to $1/\tau$ is concerned, $C^{(4)}$ (equation (3.12)), $C_p^{(4)}$ (equation (4.8)) and $C_r^{(4)}$ (equation (4.9)) are not identical and are likely to lead to slightly different predictions, especially for values of τ close to 2. It should prove useful to identify the one that provides a better approximation to the actual capacity and this task is addressed below.

5. Results and discussion

With a view to assessing their level of accuracy and range of applicability with respect to τ, the above derived analytical approximations of various orders of accuracy with respect to $1/\tau$ are compared with exact solutions of the problem. Though, ideally, it would be appropriate to make a point-wise comparison of the various orders of approximations for $\phi(\xi, \eta)$ with the exact values, it is not attempted here since, as pointed out earlier, $\phi^{(N)}(\xi, \eta)$, $N = 1, 2, \ldots$, are not amenable to closed-form evaluation and need some kind of numerical quadrature. Instead, the assessment of the approximate solutions is based on the assessment of the corresponding expressions of $C^{(N)}$, $N = 1, \ldots, 4$ (equations (3.12)–(3.16)). Further, in the following, the assessment is confined to $C^{(1)}$ and $C^{(4)}$ since these two are, respectively, the lowest and the highest order approximation at our disposal. In the case of $C^{(4)}$, all the three expressions, namely the original rational form (equation (3.12)) and its two approximations, $C_p^{(4)}$ (equation (4.8)) and $C_r^{(4)}$ (equation (4.9)), are considered.

The exact solutions were obtained by solving the linear system (2.18) numerically for progressively increasing values of N until a desired level of accuracy was achieved. Recalling the definitions of $J_{n,s}^{(N)}$ and I_n from equations (2.19) and (2.5), respectively, it can be easily shown that the coefficient matrix of the linear system is unsymmetric and diagonally dominant. Consequently, it (the linear system) can be solved without any difficulty either by iteration or by elimination techniques. In the present instance, the numerical solution of the linear system was accomplished by Gaussian elimination with scaled partial pivoting and iterative improvement [29] for selected values of $\tau \geqslant 2$. For all the results to be reported here, the numerical solutions are accurate to at least the fifth digit and may, for all practical purposes, be deemed as the exact solutions. It was noted that the convergence of the solution with N was a little slow for $\tau = 2$, requiring about 40 terms (i.e., $N = 40$) for a fifth-digit accuracy; it must, however, be noted that the case of $\tau = 2$ is somewhat special since our method is, strictly speaking, not valid at $\tau = 2$. The number of terms needed for a five-digit convergence decreases very rapidly with increasing τ and at $\tau = 3$ convergence to about eight digits was achieved with $N = 16$. It was further noted that the convergence was slightly slower in the case of oppositely charged plates. As a check on the validity of the analytical developments, the numerical solutions from our linear system (2.18) were compared with those given by Cooke [21], who obtained them by solving Love's integral equation (1.4) numerically. It was seen that the correspondence between the two sets of results was excellent. Thus, for equally charged plates, our results indicate, for $\tau = 2$, a value of 0.497 76 for C while Cooke's value is 0.497 65; the corresponding figures at $\tau = 3$ are 0.531 08 and 0.531 07, respectively. Similarly, in the case of oppositely charged plates, our results give values of 0.883 42 and 0.794 57 for C at τ equal to 2 and 3, respectively, while the corresponding Cooke's values are 0.882 80 and 0.794 75, respectively. The slightly inferior correspondence at $\tau = 2$ may be attributed to the fact that our method is not strictly valid for this τ and the accuracy of Cooke's results degrades as τ is decreased.

Shown in figure 2 are C (exact, numerical), $C^{(1)}$ (equation (3.16)) and $C^{(4)}$ (equation (3.12)) plotted against τ for equally charged plates. Figure 3 is a similar plot for oppositely charged plates. In both the figures, the exact curves are extended to values of τ lower than 2 with the help of results taken from Cooke [21]. Though, for both cases, C approaches its limiting value of $2/\pi$ asymptotically as $\tau \to \infty$, the plots are confined to $\tau = 17$ since beyond this value all the three, namely $C^{(1)}$, $C^{(4)}$ and C, give identical numbers so that the differences among them cannot be resolved adequately. It may be seen that in the case of equally charged plates the capacity is slightly overpredicted by $C^{(4)}$ and somewhat underpredicted by $C^{(1)}$, while exactly the opposite is the case with oppositely charged plates.

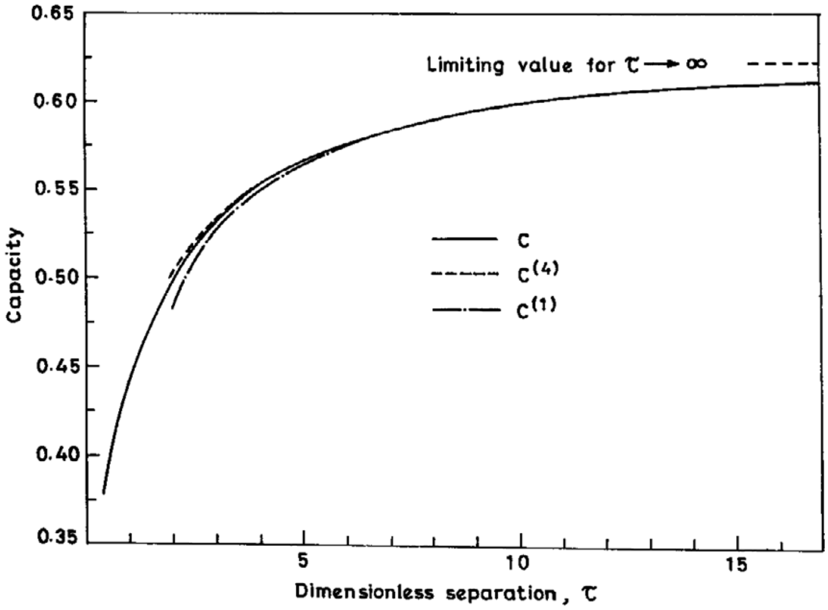

Figure 2. Variation of C, $C^{(1)}$ and $C^{(4)}$ with τ for equally charged plates.

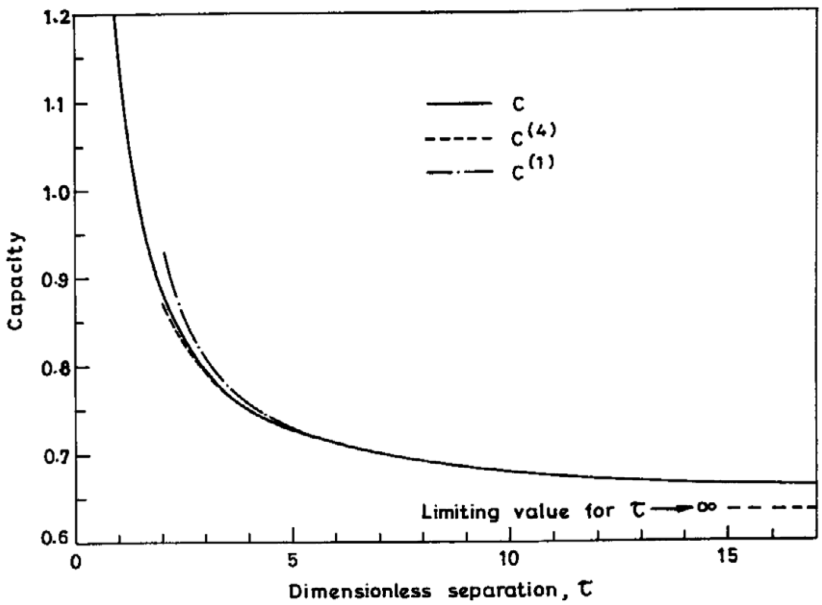

Figure 3. Variation of C, $C^{(1)}$ and $C^{(4)}$ with τ for oppositely charged plates.

However, the predictions from all the three are so close over the entire range of $\tau \geqslant 2$ that differences among them cannot be resolved adequately from a graph. Consequently, in the following, the assessment of the level of accuracy and range of applicability of both $C^{(4)}$ and $C^{(1)}$ will be based on tabulated data.

Table 1. Comparison of $C^{(1)}$, $C^{(4)}$, $C_p^{(4)}$ and $C_r^{(4)}$ with C at selected values of τ ($\geqslant$2) for equally charged plates ($\lambda = 1$).

τ	C (exact)	$C^{(1)}$ (equation (3.16))	$\delta^{(1)}$ (equation (5.1))	$C^{(4)}$ (equation (3.12))	$C_p^{(4)}$ (equation (4.8))	$\delta_p^{(4)}$ (equation (5.1))	$C_r^{(4)}$ (equation (4.9))	$\delta_r^{(4)}$ (equation (5.1))
2.0	0.497 76	0.482 91	20.984	0.499 40	0.499 17	0.284	0.499 41	0.331
2.2	0.505 79	0.493 74	20.382	0.506 59	0.506 49	0.139	0.506 59	0.158
2.4	0.513 09	0.503 15	10.936	0.513 48	0.513 44	0.070	0.573 48	0.077
2.6	0.519 68	0.511 40	10.592	0.519 88	0.519 86	0.036	0.519 88	0.040
2.8	0.525 65	0.518 69	10.324	0.525 76	0.525 75	0.019	0.525 76	0.021
3.0	0.531 08	0.525 17	10.112	0.531 14	0.531 14	0.011	0.531 14	0.012
4.0	0.552 09	0.549 21	0.522	0.552 10	0.552 10	0.001	0.552 10	0.001
5.0	0.566 32	0.564 72	0.284	0.566 32	0.566 32	0.0	0.566 32	0.0
6.0	0.576 53	0.575 55	0.170	0.576 53	0.576 53	0.0	0.576 53	0.0
7.0	0.584 19	0.583 55	0.110	0.584 19	0.584 19	0.0	0.584 19	0.0
8.0	0.590 14	0.589 69	0.075	0.590 14	0.590 14	0.0	0.590 14	0.0
9.0	0.594 88	0.594 56	0.053	0.594 88	0.594 88	0.0	0.594 88	0.0
10.0	0.598 75	0.598 52	0.039	0.598 75	0.598 75	0.0	0.598 75	0.0
15.0	0.610 77	0.610 70	0.012	0.610 77	0.610 77	0.0	0.610 77	0.0
20.0	0.617 01	0.616 98	0.005	0.617 01	0.617 01	0.0	0.617 01	0.0

Table 2. Comparison of $C^{(1)}$, $C^{(4)}$, $C_p^{(4)}$ and $C_r^{(4)}$ with C at selected values of τ ($\geqslant$ 2) for oppositely charged plates ($\lambda = -1$).

τ	C (exact)	$C^{(1)}$ (equation (3.16))	$\delta^{(1)}$ (equation (5.1))	$C^{(4)}$ (equation (3.12))	$C_p^{(4)}$ (equation (4.8))	$\delta_p^{(4)}$ (equation (5.1))	$C_r^{(4)}$ (equation (4.9))	$\delta_r^{(4)}$ (equation (5.1))
2.0	0.883 42	0.933 88	5.712	0.878 65	0.877 94	0.621	0.878 66	0.539
2.2	0.859 04	0.895 86	4.286	0.856 87	0.856 59	0.284	0.856 87	0.252
2.4	0.838 69	0.866 45	3.311	0.837 68	0.837 56	0.134	0.837 68	0.120
2.6	0.821 60	0.843 04	2.610	0.821 10	0.821 05	0.067	0.821 10	0.061
2.8	0.807 07	0.823 96	2.093	0.806 81	0.806 79	0.035	0.806 81	0.032
3.0	0.794 57	0.808 11	1.703	0.794 43	0.794 42	0.019	0.794 43	0.018
4.0	0.751 72	0.757 12	0.718	0.751 71	0.751 71	0.002	0.751 71	0.001
5.0	0.726 84	0.729 50	0.366	0.726 84	0.726 84	0.0	0.726 84	0.0
6.0	0.710 69	0.712 19	0.211	0.710 69	0.710 69	0.0	0.710 69	0.0
7.0	0.699 39	0.700 31	0.132	0.699 39	0.699 39	0.0	0.699 39	0.0
8.0	0.691 05	0.691 66	0.088	0.691 05	0.691 05	0.0	0.691 05	0.0
9.0	0.684 66	0.685 08	0.061	0.684 66	0.684 66	0.0	0.684 66	0.0
10.0	0.679 60	0.679 90	0.045	0.679 60	0.679 60	0.0	0.679 01	0.0
15.0	0.664 75	0.664 84	0.013	0.664 75	0.664 75	0.0	0.664 75	0.0
20.0	0.657 51	0.657 55	0.006	0.657 51	0.657 51	0.0	0.657 51	0.0

Presented in tables 1 and 2 are values of C (direct numerical solution), $C^{(1)}$ (equation (3.16)), $C^{(4)}$ (equation (3.12)), $C_p^{(4)}$ (equation (4.8)) and $C_r^{(4)}$ (equation (4.9)) rounded off to five significant digits for selected values of τ ranging from 2 to 20; the results shown in table 1 are for the case of equally charged plates and those of table 2 represent the case of oppositely charged plates. Also shown in these tables are the per cent relative errors in

$C^{(1)}$, $C_p^{(4)}$ and $C_r^{(4)}$ defined, successively, as

$$\delta^{(1)} = \left|1 - \frac{C^{(1)}}{C}\right| \times 100, \qquad \delta_p^{(4)} = \left|1 - \frac{C_p^{(4)}}{C}\right| \times 100$$
$$\text{and} \qquad \delta_r^{(4)} = \left|1 - \frac{C_r^{(4)}}{C}\right| \times 100. \tag{5.1}$$

Even though, as is evident from figures 2 and 3, the capacity continues to change significantly well beyond $\tau = 17$, the tabulation is limited to $\tau \leqslant 20$, since, beyond this value, the different estimates of C become almost identical. Also, no further graphical display of the results is attempted, since, as can be easily judged from tables 1 and 2, the differences among C, $C^{(1)}$, etc are rather small and cannot be adequately resolved in a graph.

Concerning the different expressions of $C^{(4)}$, it may be seen that, for both types of plates, the predictions from the approximate expression $C_r^{(4)}$ (equation (4.9)) coincide with those from the exact expression (equation (3.12)) up to at least the fourth digit for any $\tau \geqslant 2$. The other approximate expression, namely $C_p^{(4)}$ (equation (4.8)), on the other hand, leads to slightly lower values than $C^{(4)}$, the differences between the two showing up, for τ close to 2, in the third digit and beyond in the case of oppositely charged plates and from the fourth digit onwards for equally charged plates. These differences are, however, of some significance only for values of τ close to 2 and decrease rather rapidly with increasing τ so that, beyond $\tau = 4$, the predictions from all the three expressions of $C^{(4)}$ become virtually indistinguishable. It is thus apparent that the kind of simplification effected in the higher order approximations of C is perfectly justified and, accordingly, the exact expression of $C^{(4)}$ will not be discussed further.

5.1. *Equally charged plates* ($\lambda = 1$)

Referring to table 1, it may be seen that for equally charged plates, the capacity is somewhat underpredicted by $C^{(1)}$ and slightly overpredicted by both $C_p^{(4)}$ and $C_r^{(4)}$. Between the two of them, $C_p^{(4)}$ leads to slightly lower values than $C_r^{(4)}$ and thus constitutes, if only marginally, a better approximation (to C) than the latter. In the light of this observation, the following discussion will be limited mainly to $C^{(1)}$ and $C_p^{(4)}$.

From table 1, where the per cent relative errors $\delta^{(1)}$ and $\delta_p^{(4)}$ of $C^{(1)}$ and $C_p^{(4)}$, respectively, are also shown, it is apparent that $C_p^{(4)}$ predicts the capacity to within 0.1% for $\tau > 2.2$ and a similar prediction is possible with $C^{(1)}$ for $\tau > 7$. Even at the lowest value of τ, namely 2, for which the analytical approximations are valid, $C_p^{(4)}$ predicts the constriction resistance to within about 0.3%. Since accuracies better than this are rarely needed, it may be concluded that $C_p^{(4)}$, which, for this case of equally charged plates, may be expressed as

$$C_p^{(4)} = \left[1.5708 + \frac{1}{\tau} - \frac{0.666\,67}{\tau^3} + \frac{1.0667}{\tau^5} - \frac{0.056\,59}{\tau^6} - \frac{2.2857}{\tau^7} + \frac{0.323\,36}{\tau^8}\right]^{-1}, \tag{5.2}$$

is a very good approximation to C for any $\tau \geqslant 2$. Surprisingly, even $C^{(1)}$, which assumes the form

$$C^{(1)} = \left(1.5708 + \frac{1}{\tau}\right)^{-1}, \tag{5.3}$$

seems to be a fairly adequate approximation if τ is not too close to 2. As can be judged from table 1, the relative error in $C^{(1)}$ is less than 1% for $\tau \geqslant 4$, indicating that, for all practical purposes, C may be predicted quite adequately by $C^{(1)}$ for any $\tau \geqslant 4$.

5.2. Oppositely charged plates ($\lambda = -1$)

In this case (table 2), the capacity is somewhat overestimated by $C^{(1)}$ and slightly underestimated by both $C_p^{(4)}$ and $C_r^{(4)}$. Between the two of them, $C_r^{(4)}$ predicts slightly higher values than $C_p^{(4)}$ and thus makes a slightly better approximation than the latter. Accordingly, the present discussion will be devoted mainly to $C^{(1)}$ and $C_r^{(4)}$.

From table 2, where the per cent relative errors $\delta^{(1)}$ and $\delta_r^{(4)}$ of $C^{(1)}$ and $C_r^{(4)}$, respectively, are also shown, it may be seen that $C_r^{(4)}$, which assumes, for this case of oppositely charged plates, the form

$$C_r^{(4)} = \frac{\left(1 - \frac{0.339\,53}{\tau^5} + \frac{1.4551}{\tau^7}\right)}{\left(1.5708 - \frac{1}{\tau} + \frac{0.666\,67}{\tau^3} - \frac{1.6}{\tau^5} + \frac{0.282\,94}{\tau^6} + \frac{4.5714}{\tau^7} - \frac{1.3581}{\tau^8}\right)}, \tag{5.4}$$

predicts the constriction resistance to within 0.1% for $\tau > 2.4$ and a similar prediction is possible for $\tau \geqslant 8$ with $C^{(1)}$, which reduces, in this case, to

$$C^{(1)} = \left(1.5708 - \frac{1}{\tau}\right)^{-1}. \tag{5.5}$$

Even for $2 \leqslant \tau \leqslant 2.4$, $C_r^{(4)}$ seems to be a fairly accurate approximation, the relative error being not more than 0.54% in the worst case of $\tau = 2$. In a similar vein, $C^{(1)}$ may be deemed adequate for $\tau \geqslant 4$, since, as may be ascertained from table 2, in this regime the relative error remains below about 0.7%.

5.3. Assessment of Sneddon's approximate solution

As pointed out in the introduction, prior to the present study, Sneddon's solution (equation (8.1.42) of [1]) was the only analytical approximation available for predicting the capacity for large distances of separation between the discs. It is, therefore, of some interest to see how it compares with the present solution. After correcting a minor error in equation (8.1.41) of [1], this solution may be expressed as

$$C \approx \frac{2}{\pi}\left[1 + \frac{2\lambda}{\pi}\left\{\arctan\left(\frac{2}{\tau}\right) - \frac{\tau}{4}\log\left(1 + \frac{4}{\tau^2}\right)\right\}\right]^{-1}. \tag{5.6}$$

For $\tau > 2$, both $\arctan(2/\tau)$ and $\log(1 + (4/\tau^2))$ may be expanded as power series in $(2/\tau)$, leading to the expression

$$C \approx \frac{2}{\pi}\left[1 + \frac{\lambda}{\pi}\left\{\frac{2}{\tau} - \frac{4}{3\tau^3} + \frac{32}{15\tau^5} + \frac{96}{7\tau^7} + \cdots\right\}\right]^{-1}, \qquad \tau > 2. \tag{5.7}$$

Comparing this equation with equation (4.8) and following the arguments leading to the latter, it is apparent that it (equation (5.7)) is accurate to $\mathcal{O}(\tau^{-6})$ for any $\tau > 2$ and is, thus, only marginally inferior to $C_p^{(4)}$ (equation (4.8)) and $C_r^{(4)}$ (equation (4.9)). It must, however, be pointed out that equation (5.6) seems to have somewhat wider applicability than the analytical approximations presented here; as can be judged from table 3, p 243 of [1], it (i.e., equation (5.6)) predicts the capacity with reasonable accuracy down to $\tau \approx 1$. At the same time, it must also be noted that Sneddon's solution was limited to the capacity alone, since the variational approach adopted by him leads to a direct solution for the capacity for any specified charge distribution on the discs. Extension of the method to obtain an explicit solution for the electrostatic potential was not attempted and this remains a major limitation of Sneddon's work.

6. Conclusions

The main conclusions of this study are

(1) For large gaps characterized by the condition that the distance of separation between the plates is greater than or equal to the plate diameter (i.e., $\tau \geqslant 2$), the capacity of the circular plate condenser is very closely approximated by the analytical approximation $C^{(4)}$ (equation (3.12)) and its two variants $C_p^{(4)}$ (equation (5.2)) and $C_r^{(4)}$ (equation (5.4)), all with a formal accuracy of $\mathcal{O}(\tau^{-9})$. For equally charged plates, $C_p^{(4)}$ provides the best approximation with an error which decreases very steeply from a maximum value of about 0.3% at $\tau = 2$ to less than 0.1% for $\tau \geqslant 2.2$. For oppositely charged plates, on the other hand, $C_r^{(4)}$ constitutes the most accurate approximation with an error that falls off very rapidly from a maximum of about 0.54% at $\tau = 2$ to less than 0.1% for $\tau > 2.4$. Though not actually tested, it may be surmised that the electrostatic potential may be predicted with similar accuracy by the $\mathcal{O}(\tau^{-9})$ analytical approximation.

(2) For both types of plates, the capacity may be predicted with adequate accuracy (error $<$1.0%) even by the lowest order analytical approximation $C^{(1)}$ (equations (5.3) and (5.5)), which is accurate only to $\mathcal{O}(\tau^{-3})$, if $\tau > 4$.

(3) An analytical approximation for the capacity due to Sneddon [1], which is based on the assumption that, for large distances of separation, the charge distribution on either of the plates is the same as that existing on an isolated plate raised to a uniform potential, is accurate to $\mathcal{O}(\tau^{-6})$ for any $\tau \geqslant 2$.

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